

Continuity of Minima: Local Results

Eugene A. Feinberg · Pavlo O. Kasyanov

Received: date / Accepted: date

Abstract This paper compares and generalizes Berge's maximum theorem for noncompact image sets established in Feinberg, Kasyanov and Voorneveld [5] and the local maximum theorem established in Bonnans and Shapiro [3, Proposition 4.4].

Keywords Berge's maximum theorem · Set-valued mapping · Continuity

PACS 02.30.Xx · 02.30.Yy · 02.30.Sa

Mathematics Subject Classification (2010) MSC 49J27 · 49J45

1 Introduction and Main Results

Let \mathbb{X} and \mathbb{Y} be Hausdorff topological spaces, and $\Phi : \mathbb{X} \rightarrow \mathbb{S}(\mathbb{Y})$ be a set-valued map, where $\mathbb{S}(\mathbb{Y}) := 2^{\mathbb{Y}} \setminus \{\emptyset\}$ is the family of all nonempty subsets of the set \mathbb{Y} . Consider the graph of Φ , defined as $\text{Gr}_{\mathbb{X}}(\Phi) = \{(x, y) \in \mathbb{X} \times \mathbb{Y} : y \in \Phi(x)\}$, and let $u : \text{Gr}_{\mathbb{X}}(\Phi) \subseteq \mathbb{X} \times \mathbb{Y} \rightarrow \overline{\mathbb{R}}$, where $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ is the extended real line. Define the value function

$$v(x) := \inf_{y \in \Phi(x)} u(x, y), \quad x \in \mathbb{X}, \quad (1)$$

This research was partially supported by NSF grant CMMI-1335296, by the Ukrainian State Fund for Fundamental Research under grant GP/F49/070, and by grant 2273/14 from the National Academy of Sciences of Ukraine.

E.A. Feinberg

Department of Applied Mathematics and Statistics, Stony Brook University, Stony Brook, NY 11794-3600, USA,

Tel.: +1-631-632-7189

E-mail: eugene.feinberg@stonybrook.edu

P.O. Kasyanov

Institute for Applied System Analysis, National Technical University of Ukraine "Kyiv Polytechnic Institute", Peremogy ave., 37, build, 35, 03056, Kyiv, Ukraine, kasyanov@i.ua.

and the solution multifunction

$$\Phi^*(x) := \{y \in \Phi(x) : v(x) = u(x, y)\}, \quad x \in \mathbb{X}.$$

To clarify the above definitions, consider a Hausdorff topological space \mathbb{A} . For a nonempty set $A \subseteq \mathbb{A}$, the notation $f : A \subseteq \mathbb{A} \rightarrow \overline{\mathbb{R}}$ means that for each $a \in A$ the value $f(a) \in \overline{\mathbb{R}}$ is defined. In general, the function f may be also defined outside of A . The notation $f : \mathbb{A} \rightarrow \overline{\mathbb{R}}$ means that the function f is defined on the entire space \mathbb{A} . This notation is equivalent to the notation $f : \mathbb{A} \subseteq \mathbb{A} \rightarrow \overline{\mathbb{R}}$, which we do not write explicitly. For a function $f : A \subseteq \mathbb{A} \rightarrow \overline{\mathbb{R}}$ we sometimes consider its restriction $f : B \subseteq \mathbb{A} \rightarrow \overline{\mathbb{R}}$ to the set $B \subseteq A$. Sometimes we consider functions with values in \mathbb{R} rather than in $\overline{\mathbb{R}}$.

We recall that, for a nonempty set $A \subseteq \mathbb{A}$, a function $f : A \subseteq \mathbb{A} \rightarrow \overline{\mathbb{R}}$ is called *lower semi-continuous at* $a \in A$, if for each net $\{a_i\}_{i \in I} \subset A$, that converges to a in \mathbb{A} , the inequality $\liminf_i f(a_i) \geq f(a)$ holds. A function $f : A \subseteq \mathbb{A} \rightarrow \overline{\mathbb{R}}$ is called *upper semi-continuous at* $a \in A$, if $-f$ is lower semi-continuous at $a \in A$. Consider the level sets

$$\mathcal{D}_f(\lambda; A) := \{a \in A : f(a) \leq \lambda\}, \quad \lambda \in \mathbb{R}.$$

A function $f : A \subseteq \mathbb{A} \rightarrow \overline{\mathbb{R}}$ is called *lower semi-continuous on* A , if all the level sets $\mathcal{D}_f(\lambda; A)$ are closed in \mathbb{A} . A function $f : A \subseteq \mathbb{A} \rightarrow \overline{\mathbb{R}}$ is called *inf-compact* (such a function is sometimes called *lower semi-compact*) *on* A , if all the level sets are compact in \mathbb{A} . We notice that, if $A = \mathbb{A}$ and $f : \mathbb{A} \rightarrow \overline{\mathbb{R}}$, then f is lower semi-continuous at each $a \in \mathbb{A}$ if and only if it is lower semi-continuous on \mathbb{A} . For an arbitrary nonempty subset A of \mathbb{A} , lower semi-continuity on A implies lower semi-continuity at each $a \in A$, but not vice versa. Indeed, if $A \subseteq \mathbb{A}$ is a nonempty set, $f : A \subseteq \mathbb{A} \rightarrow \overline{\mathbb{R}}$ is lower semi-continuous on A , and $\{a_i\}_{i \in I} \subset A$ converges to $a \in A$, then either $\liminf_i f(a_i) = +\infty$ or there exists a subnet $\{a_j\}_{j \in J} \subseteq \{a_i\}_{i \in I}$ such that $\{a_j\}_{j \in J}$ is eventually in $\mathcal{D}_f(\lambda; A)$ for each real $\lambda > \liminf_i f(a_i)$. Thus $f(a) \leq \liminf_i f(a_i)$. Vice versa, if $\mathbb{A} = [0, 1]$, $A = (0, 1]$, and $f \equiv 0$ on A , then f is lower semi-continuous at each $a \in A$, but it is not lower semi-continuous on A because $\mathcal{D}_f(1; A) = A$ is not closed in \mathbb{A} .

Definition 1 (cf. Feinberg et al. [5, Definition 1.3]). A function $u : \text{Gr}_{\mathbb{X}}(\Phi) \subseteq \mathbb{X} \times \mathbb{Y} \rightarrow \overline{\mathbb{R}}$ is called \mathbb{KN} -inf-compact on $\text{Gr}_{\mathbb{X}}(\Phi)$, if the following two conditions hold:

- (i) $u : \text{Gr}_{\mathbb{X}}(\Phi) \subseteq \mathbb{X} \times \mathbb{Y} \rightarrow \overline{\mathbb{R}}$ is lower semi-continuous on $\text{Gr}_{\mathbb{X}}(\Phi)$;
- (ii) for any convergent net $\{x_i\}_{i \in I}$ with values in \mathbb{X} whose limit x belongs to \mathbb{X} , any net $\{y_i\}_{i \in I}$, defined on the same ordered set I with $y_i \in \Phi(x_i)$, $i \in I$, and satisfying the condition that the set $\{u(x_i, y_i) : i \in I\}$ is bounded above, has an accumulation point $y \in \Phi(x)$.

We remark that this definition is consistent with Feinberg et al. [5, Definition 1.3], according to which a function $u : \mathbb{X} \times \mathbb{Y} \rightarrow \overline{\mathbb{R}}$ is called \mathbb{KN} -inf-compact

on $\text{Gr}_{\mathbb{X}}(\Phi)$, if it satisfies properties (i) and (ii) of Definition 1. A function $u : \mathbb{X} \times \mathbb{Y} \rightarrow \overline{\mathbb{R}}$ is \mathbb{KN} -inf-compact on $\text{Gr}_{\mathbb{X}}(\Phi)$ in the sense of [5, Definition 1.3] if and only if its restriction $\text{Gr}_{\mathbb{X}}(\Phi)$ is \mathbb{KN} -inf-compact on $\text{Gr}_{\mathbb{X}}(\Phi)$ in the sense of Definition 1. This is true because, if u is also defined outside of $\text{Gr}_{\mathbb{X}}(\Phi)$, its values at $(x, y) \notin \text{Gr}_{\mathbb{X}}(\Phi)$ do not affect the \mathbb{KN} -inf-compactness of u on $\text{Gr}_{\mathbb{X}}(\Phi)$.

For $Z \subseteq \mathbb{X}$ define the graph of a set-valued mapping $\Phi : \mathbb{X} \rightarrow \mathbb{S}(\mathbb{Y})$, restricted to Z :

$$\text{Gr}_Z(\Phi) = \{(x, y) \in Z \times \mathbb{Y} : y \in \Phi(x)\}.$$

The following definition introduces the notion of \mathbb{KN} -inf-compactness in a local formulation.

Definition 2 Let $Z \subseteq \mathbb{X}$ be a nonempty set. A function $u : \text{Gr}_{\mathbb{X}}(\Phi) \subseteq \mathbb{X} \times \mathbb{Y} \rightarrow \overline{\mathbb{R}}$ is called \mathbb{KN} -inf-compact on $\text{Gr}_Z(\Phi)$, if the following two conditions hold:

- (i) $u : \text{Gr}_{\mathbb{X}}(\Phi) \subseteq \mathbb{X} \times \mathbb{Y} \rightarrow \overline{\mathbb{R}}$ is lower semi-continuous at all $(x, y) \in \text{Gr}_Z(\Phi)$;
- (ii) if a net $\{x_i\}_{i \in I}$ with values in \mathbb{X} converges to $x \in Z$, then each net $\{y_i\}_{i \in I}$, defined on the same ordered set I with $y_i \in \Phi(x_i)$, $i \in I$, and satisfying the condition that the set $\{u(x_i, y_i) : i \in I\}$ is bounded above, has an accumulation point $y \in \Phi(x)$.

Remark 1 If $Z = \mathbb{X}$, then Definitions 1 and 2 of \mathbb{KN} -inf-compactness on $\text{Gr}_{\mathbb{X}}(\Phi)$ are equivalent. This follows from the following statements: (a) conditions (ii) of Definitions 1 and 2 coincide, if $Z = \mathbb{X}$; (b) conditions (i) and (ii) of Definition 2 with $Z = \mathbb{X}$ imply condition (i) of Definition 1; and (c) condition (i) of Definition 1 yields condition (i) of Definition 2. Note that statement (b) holds because, if $\lambda \in \mathbb{R}$ and a net $\{(x_i, y_i)\}_{i \in I} \subset \mathcal{D}_u(\lambda; \text{Gr}_{\mathbb{X}}(\Phi))$ converges to $(x, y) \in \mathbb{X} \times \mathbb{Y}$, then condition (ii) of Definition 2 yields that $y \in \Phi(x)$ and condition (i) of Definition 2 implies that $(x, y) \in \mathcal{D}_u(\lambda; \text{Gr}_{\mathbb{X}}(\Phi))$. Therefore, the level set $\mathcal{D}_u(\lambda; \text{Gr}_{\mathbb{X}}(\Phi))$ is closed for each $\lambda \in \mathbb{R}$. Statement (c) holds, because lower semi-continuity on $\text{Gr}_{\mathbb{X}}(\Phi)$ implies lower semi-continuity at each $(x, y) \in \text{Gr}_{\mathbb{X}}(\Phi)$.

Remark 2 If spaces \mathbb{X} and \mathbb{Y} are metrizable, then nets and subnets in the definition of \mathbb{KN} -inf-compactness on $\text{Gr}_Z(\Phi)$ can be replaced with sequences and subsequences; see Lemma 2.

Note that a function $u : \text{Gr}_{\mathbb{X}}(\Phi) \subseteq \mathbb{X} \times \mathbb{Y} \rightarrow \overline{\mathbb{R}}$ is \mathbb{KN} -inf-compact on $\text{Gr}_Z(\Phi)$, where Z is a nonempty subset of \mathbb{X} , if and only if it is \mathbb{KN} -inf-compact on $\text{Gr}_{\{x\}}(\Phi)$ for all $x \in Z$.

For a Hausdorff topological space \mathbb{A} , we denote by $\mathbb{K}(\mathbb{A})$ the family of all nonempty compact subsets of \mathbb{A} . A function $u : \text{Gr}_{\mathbb{X}}(\Phi) \subseteq \mathbb{X} \times \mathbb{Y} \rightarrow \overline{\mathbb{R}}$ is called \mathbb{K} -inf-compact on $\text{Gr}_{\mathbb{X}}(\Phi)$, if for every $K \in \mathbb{K}(\mathbb{X})$ this function is inf-compact on $\text{Gr}_K(\Phi)$; cf. Feinberg et al. [7, Definition 1.1]. \mathbb{KN} -inf-compactness on $\text{Gr}_{\mathbb{X}}(\Phi)$ is a more restrictive property than \mathbb{K} -inf-compactness property on $\text{Gr}_{\mathbb{X}}(\Phi)$; see Feinberg et al. [5, Theorem 2.1 and Example 5.1]. As shown in

Feinberg et al. [5, Corollary 2.2], \mathbb{K} -inf-compactness and \mathbb{KN} -inf-compactness on $\text{Gr}_{\mathbb{X}}(\Phi)$ are equivalent, if \mathbb{X} is a compactly generated topological space, and, in particular, if \mathbb{X} is a metrizable topological space. Recall that a topological space \mathbb{X} is *compactly generated* (Munkres [11, p. 283] or a k -space, Kelley [10, p. 230], Engelking [4, p. 152]) if it satisfies the following property: each set $A \subseteq \mathbb{X}$ is closed in \mathbb{X} if $A \cap K$ is closed in K for each $K \in \mathbb{K}(\mathbb{X})$. In particular, all locally compact spaces (hence, manifolds) and all sequential spaces (hence, first-countable, including metrizable/metric spaces) are compactly generated; Munkres [11, Lemma 46.3, p. 283], Engelking [4, Theorem 3.3.20, p. 152].

For a set-valued mapping $F : \mathbb{X} \rightarrow 2^{\mathbb{Y}}$, let $\text{Dom}F := \{x \in \mathbb{X} : F(x) \neq \emptyset\}$. Recall that a set-valued mapping $F : \mathbb{X} \rightarrow 2^{\mathbb{Y}}$ is *upper semi-continuous* at $x \in \text{Dom}F$, if, for each neighborhood \mathcal{G} of the set $F(x)$, there is a neighborhood of x , say $\mathcal{O}(x)$, such that $F(x^*) \subseteq \mathcal{G}$ for all $x^* \in \mathcal{O}(x)$. A set-valued mapping $F : \mathbb{X} \rightarrow 2^{\mathbb{Y}}$ is *lower semi-continuous* at $x \in \text{Dom}F$, if, for each open set \mathcal{G} with $F(x) \cap \mathcal{G} \neq \emptyset$, there is a neighborhood of x , say $\mathcal{O}(x)$, such that if $x^* \in \mathcal{O}(x) \cap \text{Dom}F$, then $F(x^*) \cap \mathcal{G} \neq \emptyset$. A set-valued mapping $F : \mathbb{X} \rightarrow \mathbb{S}(\mathbb{Y})$ is called *upper (lower) semi-continuous*, if it is upper (lower) semi-continuous at all $x \in \mathbb{X}$. A set-valued mapping $F : \mathbb{X} \rightarrow \mathbb{S}(\mathbb{Y})$ is called *continuous*, if it is upper and lower semi-continuous. A set-valued mapping $F : \mathbb{X} \rightarrow \mathbb{S}(\mathbb{Y})$ is *closed*, if $\text{Gr}_{\mathbb{X}}(F)$ is a closed subset of $\mathbb{X} \times \mathbb{Y}$.

For Hausdorff topological spaces, Berge's maximum theorem for noncompact image sets has the following formulation.

Theorem 1 (Berge's maximum theorem for noncompact image sets; Feinberg et al. [5, Theorem 1.4]). *If a function $u : \mathbb{X} \times \mathbb{Y} \rightarrow \mathbb{R}$ is \mathbb{KN} -inf-compact and upper semi-continuous on $\text{Gr}_{\mathbb{X}}(\Phi)$ and $\Phi : \mathbb{X} \rightarrow \mathbb{S}(\mathbb{Y})$ is a lower semi-continuous set-valued mapping, then the value function $v : \mathbb{X} \rightarrow \mathbb{R}$ is continuous and the solution multifunction $\Phi^* : \mathbb{X} \rightarrow \mathbb{K}(\mathbb{Y})$ is upper semi-continuous and compact-valued.*

In the classic Berge's maximum theorem [2, p. 116], the function u is assumed to be continuous and the set-valued mapping Φ is assumed to be continuous and compact-valued. As explained in Feinberg et al. [5], these assumptions are more restrictive than the assumptions of Theorem 1.

Theorem 1 yields from the following three statements:

- (B1) (Lower semi-continuity of minima). If a function $u : \mathbb{X} \times \mathbb{Y} \rightarrow \overline{\mathbb{R}}$ is \mathbb{KN} -inf-compact on $\text{Gr}_{\mathbb{X}}(\Phi)$, then the value function $v : \mathbb{X} \rightarrow \overline{\mathbb{R}}$ is lower semi-continuous; Feinberg et al. [5, Theorem 3.4].
- (B2) (Upper semi-continuity of minima). If a function $u : \mathbb{X} \times \mathbb{Y} \rightarrow \overline{\mathbb{R}}$ is upper semi-continuous on $\text{Gr}_{\mathbb{X}}(\Phi)$ and $\Phi : \mathbb{X} \rightarrow \mathbb{S}(\mathbb{Y})$ is a lower semi-continuous set-valued mapping, then the value function $v : \mathbb{X} \rightarrow \overline{\mathbb{R}}$ is upper semi-continuous; Hu and Papageorgiou [9, Proposition 3.1, p. 82].
- (B3) (Upper semi-continuity of the solution multifunction). If a function $u : \mathbb{X} \times \mathbb{Y} \rightarrow \mathbb{R}$ is \mathbb{KN} -inf-compact on $\text{Gr}_{\mathbb{X}}(\Phi)$ and the value function $v : \mathbb{X} \rightarrow \mathbb{R}$ is continuous, then the solution multifunction $\Phi^* : \mathbb{X} \rightarrow \mathbb{K}(\mathbb{Y})$ is upper semi-continuous and compact-valued; Feinberg et al. [5, p. 1045].

Continuity and semi-continuity of functions and multifunctions are local properties. Therefore, it is natural to formulate the results on continuity of value functions and solution multifunctions in a local form. This is done in Bonnans and Shapiro [3, Proposition 4.4].

Theorem 2 (Bonnans and Shapiro [3, Proposition 4.4]). *Let $u : \mathbb{X} \times \mathbb{Y} \rightarrow \mathbb{R}$ and $x \in \mathbb{X}$. Supposed that:*

- (i) *the function u is continuous on $\mathbb{X} \times \mathbb{Y}$;*
- (ii) *the set-valued mapping $\Phi : \mathbb{X} \rightarrow \mathbb{S}(\mathbb{Y})$ is closed;*
- (iii) *there exists $\lambda \in \mathbb{R}$ and a compact set $C \subset \mathbb{Y}$ such that, for every x^* in a neighborhood of x , the level set $\mathcal{D}_{u(x^*, \cdot)}(\lambda; \Phi(x^*))$ is nonempty and contained in C ;*
- (iv) *for each neighborhood $\mathcal{O}(\Phi^*(x))$ of the set $\Phi^*(x)$, there exists a neighborhood $\mathcal{O}(x)$ of x such that $\mathcal{O}(\Phi^*(x)) \cap \Phi(x^*) \neq \emptyset$ for all $x^* \in \mathcal{O}(x)$.*

Then:

- (a) *the function v is continuous at x ,*
- (b) *the set-valued function Φ^* is upper semi-continuous at x .*

As Examples 1 and 2 demonstrate, Theorem 1 and Theorem 2 do not imply each other. Theorem 7 below generalizes the both theorems. In order to clarify the relevance between Theorems 1 and 2, similarly to Theorem 1, it is natural to divide Theorem 2 into the following three statements:

- (BS1) (Lower semi-continuity of minima). If the function $u : \mathbb{X} \times \mathbb{Y} \rightarrow \mathbb{R}$ is lower semi-continuous on $\mathbb{X} \times \mathbb{Y}$, $x \in \mathbb{X}$, and assumptions (ii) and (iii) of Theorem 2 hold, then the value function v is lower semi-continuous at x ; Bonnans and Shapiro [3, pp. 290–291].
- (BS2) (Upper semi-continuity of minima). If a function $u : \mathbb{X} \times \mathbb{Y} \rightarrow \mathbb{R}$ is upper semi-continuous on $\mathbb{X} \times \mathbb{Y}$, $x \in \mathbb{X}$, the set $\Phi^*(x)$ is compact, and assumption (iv) of Theorem 2 holds, then the value function v is upper semi-continuous at x ; Bonnans and Shapiro [3, p. 291].
- (BS3) (Upper semi-continuity of the solution multifunction). Let $u : \mathbb{X} \times \mathbb{Y} \rightarrow \mathbb{R}$ and $x \in \mathbb{X}$. If assumptions (i)–(iv) of Theorem 2 hold, then the solution multifunction $\Phi^* : \mathbb{X} \rightarrow \mathbb{S}(\mathbb{Y})$ is upper semi-continuous at x ; Bonnans and Shapiro [3, pp. 291].

Statement (BS1) can be derived from statement (B1) in the following way. Let assumptions of statement (BS1) hold. Consider a new Hausdorff state space $\tilde{\mathbb{X}}$ that equals to the neighborhood of x from assumption (iii) of Theorem 2 endowed with the induced topology. Let $\tilde{\mathbb{Y}} = \mathbb{Y}$. Consider new image sets $\tilde{\Phi}(z) = \mathcal{D}_{u(z, \cdot)}(\lambda; \Phi(z))$, $z \in \tilde{\mathbb{X}}$, which are nonempty and contained to the same compact subset C of $\tilde{\mathbb{Y}}$. Define the function \tilde{u} as the restriction of u to $\text{Gr}(\tilde{\Phi})$. Assumptions (ii) and (iii) of Theorem 2, Bonnans and Shapiro [3, Lemma 4.3], lower semi-continuity of \tilde{u} on $\tilde{\mathbb{X}} \times \tilde{\mathbb{Y}}$, and Feinberg et al. [5, Lemma 3.3(i)] imply \mathbb{KN} -inf-compactness of \tilde{u} on $\text{Gr}_{\tilde{\mathbb{X}}}(\tilde{\Phi})$. Therefore, statement (B1) yields lower semi-continuity of v at x . Statements (B2) and (BS2) (also (B3) and (BS3) respectively) do not imply each other; see Examples 3–5.

In the rest of this section we formulate the main results of this paper. The following theorem is more general than statement (B1), and therefore statement (BS1) follows from it.

Theorem 3 (Lower semi-continuity of minima). *Let $u : \text{Gr}_{\mathbb{X}}(\Phi) \subseteq \mathbb{X} \times \mathbb{Y} \rightarrow \overline{\mathbb{R}}$ and $x \in \mathbb{X}$. If the function u is \mathbb{KN} -inf-compact on $\text{Gr}_{\{x\}}(\Phi)$, then the function $v : \mathbb{X} \rightarrow \overline{\mathbb{R}}$ is lower semi-continuous at x and $\Phi^*(x)$ is a nonempty compact set, if $v(x) < +\infty$, and $\Phi^*(x) = \emptyset$ otherwise.*

Observe that under conditions of Theorem 3, the infimum in (1) can be replaced with the minimum. The following theorem generalizes statements (B2) and (BS2).

Theorem 4 (Upper semi-continuity of minima). *Let $u : \text{Gr}_{\mathbb{X}}(\Phi) \subseteq \mathbb{X} \times \mathbb{Y} \rightarrow \overline{\mathbb{R}}$ and $x \in \mathbb{X}$. Each of the following assumptions:*

- (i) *the function $u : \text{Gr}_{\mathbb{X}}(\Phi) \subseteq \mathbb{X} \times \mathbb{Y} \rightarrow \overline{\mathbb{R}}$ is upper semi-continuous at all $(x, y) \in \text{Gr}_{\{x\}}(\Phi)$ and $\Phi : \mathbb{X} \rightarrow \mathbb{S}(\mathbb{Y})$ is a lower semi-continuous set-valued mapping at x ;*
- (ii) *$\Phi^*(x) \in \mathbb{K}(\mathbb{Y})$, the function $u : \text{Gr}_{\mathbb{X}}(\Phi) \subseteq \mathbb{X} \times \mathbb{Y} \rightarrow \overline{\mathbb{R}}$ is upper semi-continuous at all $(x, y) \in \text{Gr}_{\{x\}}(\Phi^*)$, and assumption (iv) of Theorem 2 holds;*

implies that the function $v : \mathbb{X} \rightarrow \overline{\mathbb{R}}$ is upper semi-continuous at x .

According to Example 3, assumptions (i) and (ii) of Theorem 4 do not imply each other. Assumption (i) implies assumption (ii) in Theorem 4, if $\Phi^*(x) \in \mathbb{K}(\mathbb{Y})$ because lower semi-continuity of Φ at x with $\Phi^*(x) \in \mathbb{K}(\mathbb{Y})$ yields assumption (iv) of Theorem 2. Moreover, if $\Phi^*(x) \in \mathbb{K}(\mathbb{Y})$, then assumption (iv) of Theorem 4 follows from upper semi-continuity as well as from lower semi-continuity of Φ^* at x . The following theorem generalizes statement (B3).

Theorem 5 (Upper semi-continuity of the solution multifunction). *Let $u : \text{Gr}_{\mathbb{X}}(\Phi) \subseteq \mathbb{X} \times \mathbb{Y} \rightarrow \overline{\mathbb{R}}$ and $x \in \mathbb{X}$. If u is \mathbb{KN} -inf-compact on $\text{Gr}_{\{x\}}(\Phi)$, the value function $v : \mathbb{X} \rightarrow \overline{\mathbb{R}}$ is continuous at x , and $v(x) < +\infty$, then $\Phi^*(x) \in \mathbb{K}(\mathbb{Y})$ and Φ^* is upper semi-continuous at x .*

Being combined, Theorems 3 and 4 provide sufficient conditions for the continuity of v at x . Note that \mathbb{KN} -inf-compactness of u on $\text{Gr}_{\{x\}}(\Phi)$ in Theorem 5 cannot be weakened to lower semi-continuity of u on $\text{Gr}_{\mathbb{X}}(\Phi)$, inf-compactness of $u(x, \cdot)$ on $\Phi(x)$, and the assumptions that $\Phi : \mathbb{X} \rightarrow \mathbb{K}(\mathbb{Y})$ and $\text{Gr}_{\mathbb{X}}(\Phi)$ is closed in $\mathbb{X} \times \mathbb{Y}$; see Example 6. We also remark that the continuity of v at x is the essential assumption in Theorem 5; see Example 7.

Theorem 6 (Local optimum theorem). *Let $u : \text{Gr}_{\mathbb{X}}(\Phi) \subseteq \mathbb{X} \times \mathbb{Y} \rightarrow \overline{\mathbb{R}}$ and $x \in \mathbb{X}$ satisfy the following properties:*

- (a) *$u(x, y) < +\infty$ for some $y \in \Phi(x)$;*
- (b) *u is \mathbb{KN} -inf-compact on $\text{Gr}_{\{x\}}(\Phi)$;*

then $\Phi^*(x) \in \mathbb{K}(\mathbb{Y})$. If, in addition, $u : \text{Gr}_{\mathbb{X}}(\Phi) \subseteq \mathbb{X} \times \mathbb{Y} \rightarrow \overline{\mathbb{R}}$ is upper semi-continuous at all $(x, y) \in \text{Gr}_{\{x\}}(\Phi^*)$, then the following two assumptions are equivalent:

- (i) assumption (iv) of Theorem 2;
- (ii) the solution multifunction $\Phi^* : \mathbb{X} \rightarrow 2^{\mathbb{Y}}$ is upper semi-continuous at x ;

and each of them implies that v is continuous at x .

Observe that assumptions of Theorem 6 include conditions on $y^* \in \Phi(x^*)$, when $u(x^*, y^*) \geq \lambda > v(x)$, $x^* \in \mathcal{O}(x)$, where $\mathcal{O}(x)$ is some neighborhood of x . These values of y^* do not affect the properties of v and Φ^* at x . In order to obtain a more delicate result, we introduce the sets $\Phi_{\lambda, x}$ and function $u_{\lambda, x}$.

For $\lambda \in \mathbb{R}$, $x \in \mathbb{X}$, and $\Phi : \mathbb{X} \rightarrow \mathbb{S}(\mathbb{Y})$, define $\Phi_{\lambda, x} : \mathbb{X} \rightarrow 2^{\mathbb{Y}}$,

$$\Phi_{\lambda, x}(z) := \begin{cases} \{y \in \Phi(z) : u(z, y) \leq \lambda\}, & \text{if either this set is not empty or } z = x; \\ \Phi_{\lambda, x}(x), & \text{otherwise.} \end{cases}$$

As follows from this definition, $\Phi_{\lambda, x}(x) = \emptyset$ if and only if $u(x, y) > \lambda$ for all $y \in \Phi(x)$. In particular, if $\lambda < v(x)$, then $\Phi_{\lambda, x}(x) = \emptyset$. We remark that $\Phi_{\lambda, x} : \mathbb{X} \rightarrow \mathbb{S}(\mathbb{Y})$ if and only if $\Phi_{\lambda, x}(x) \neq \emptyset$. If $\Phi_{\lambda, x}(x) \neq \emptyset$, define the functions $u_{\lambda, x} : \mathbb{X} \times \mathbb{Y} \rightarrow \mathbb{R}$,

$$u_{\lambda, x}(z, y) := \begin{cases} u(z, y), & \text{if } \{y \in \Phi(z) : u(z, y) \leq \lambda\} \neq \emptyset; \\ u(x, y), & \text{otherwise;} \end{cases}$$

for each $z \in \mathbb{X}$ and $y \in \mathbb{Y}$, and $v_{\lambda, x} : \mathbb{X} \rightarrow \mathbb{R}$,

$$v_{\lambda, x}(z) := \inf_{y \in \Phi_{\lambda, x}(z)} u_{\lambda, x}(z, y), \quad z \in \mathbb{X}.$$

Consider the solution multifunction

$$\Phi_{\lambda, x}^*(z) := \{y \in \Phi_{\lambda, x}(z) : v_{\lambda, x}(z) = u_{\lambda, x}(z, y)\}, \quad z \in \mathbb{X}.$$

If $\lambda > v(x)$, then $v_{\lambda, x}(x) = v(x)$ and $\Phi_{\lambda, x}^*(x) = \Phi^*(x)$. If the conditions of Theorem 3 hold, the latter is true for $\lambda \geq v(x)$. Observe that, under condition (iii) of Theorem 2, $v_{\lambda, x}(x^*) = v(x^*)$ and $\Phi_{\lambda, x}^*(x^*) = \Phi^*(x^*)$ for all x^* in a neighborhood of x .

We remark that, if u is \mathbb{KN} -inf-compact on $\text{Gr}_{\{x\}}(\Phi)$, then $u_{\lambda, x}$ is \mathbb{KN} -inf-compact on $\text{Gr}_{\{x\}}(\Phi_{\lambda, x})$. The inverse claim does not hold in general; see Example 1. The following corollary from Theorem 3 generalizes statements (B1) and (BS1).

Corollary 1 (Lower semi-continuity of minima). *Let $u : \text{Gr}_{\mathbb{X}}(\Phi) \subseteq \mathbb{X} \times \mathbb{Y} \rightarrow \overline{\mathbb{R}}$, $x \in \mathbb{X}$, and $\lambda \in \mathbb{R}$ satisfy $u(x, y) \leq \lambda$ for some $y \in \Phi(x)$. If the function $u_{\lambda, x} : \text{Gr}_{\mathbb{X}}(\Phi_{\lambda, x}) \subseteq \mathbb{X} \times \mathbb{Y} \rightarrow \overline{\mathbb{R}}$ is \mathbb{KN} -inf-compact on $\text{Gr}_{\{x\}}(\Phi_{\lambda, x})$, then the function $v : \mathbb{X} \rightarrow \overline{\mathbb{R}}$ is lower semi-continuous at x and $\Phi^*(x) \in \mathbb{K}(\mathbb{Y})$.*

The following theorem generalizes Theorems 1, 2, and 6.

Theorem 7 (Local optimum theorem). *Let $u : \text{Gr}_{\mathbb{X}}(\Phi) \subseteq \mathbb{X} \times \mathbb{Y} \rightarrow \overline{\mathbb{R}}$, $x \in \mathbb{X}$, and $\lambda \in \mathbb{R}$ satisfy $u(x, y) < \lambda$ for some $y \in \Phi(x)$. If the function $u_{\lambda, x} : \text{Gr}_{\mathbb{X}}(\Phi_{\lambda, x}) \subseteq \mathbb{X} \times \mathbb{Y} \rightarrow \overline{\mathbb{R}}$ is \mathbb{KN} -inf-compact on $\text{Gr}_{\{x\}}(\Phi_{\lambda, x})$ and the function $u : \text{Gr}_{\mathbb{X}}(\Phi) \subseteq \mathbb{X} \times \mathbb{Y} \rightarrow \overline{\mathbb{R}}$ is upper semi-continuous at all $(x, y) \in \text{Gr}_{\{x\}}(\Phi^*)$, then the following two assumptions are equivalent:*

- (i) *assumption (iv) of Theorem 2;*
- (ii) *the solution multifunction $\Phi^* : \mathbb{X} \rightarrow 2^{\mathbb{Y}}$ is upper semi-continuous at x ;*

and each of them implies that v is continuous at x .

Remark 3 Upper semi-continuity of $u : \text{Gr}_{\mathbb{X}}(\Phi) \subseteq \mathbb{X} \times \mathbb{Y} \rightarrow \overline{\mathbb{R}}$ at all $(x, y) \in \text{Gr}_{\{x\}}(\Phi^*)$ in Theorem 7 cannot be relaxed to upper semi-continuity of $u_{\lambda, x} : \text{Gr}_{\mathbb{X}}(\Phi_{\lambda, x}) \subseteq \mathbb{X} \times \mathbb{Y} \rightarrow \overline{\mathbb{R}}$ at all $(x, y) \in \text{Gr}_{\{x\}}(\Phi^*)$; see Example 9.

Remark 4 Theorems 2, 4, 6, and 7 have the common assumption: condition (iv) of Theorem 2. The remaining assumptions are weaker in Theorem 7 than in Theorem 2. Indeed, conditions (ii) and (iii) of Theorem 2 imply that $\Phi_{\lambda, x}$ is a closed mapping that acts from a closed neighborhood of x , say $\overline{\mathcal{O}}(x)$, into the compact set C . Therefore, Berge [2, Corollary, p. 112] yields that $\Phi_{\lambda, x} : \overline{\mathcal{O}}(x) \rightarrow \mathbb{S}(\mathbb{Y})$ is upper semi-continuous, compact-valued, $v_{\lambda, x}(x^*) = v(x^*)$ and $\Phi_{\lambda, x}^*(x^*) = \Phi^*(x^*)$ for all x^* in a neighborhood of x . Moreover, condition (i) of Theorem 2 and Lemma 1 imply that the function $u_{\lambda, x} : \text{Gr}_{\overline{\mathcal{O}}(x)}(\Phi_{\lambda, x}) \subseteq \mathbb{X} \times \mathbb{Y} \rightarrow \overline{\mathbb{R}}$ is \mathbb{KN} -inf-compact on $\text{Gr}_{\{x\}}(\Phi_{\lambda, x})$ and the function $u : \text{Gr}_{\overline{\mathcal{O}}(x)}(\Phi) \subseteq \mathbb{X} \times \mathbb{Y} \rightarrow \overline{\mathbb{R}}$ is upper semi-continuous at all $(x, y) \in \text{Gr}_{\{x\}}(\Phi^*)$.

2 Local Properties of \mathbb{KN} -inf-compact Functions

Lemma 1 *Let $Z \subseteq \mathbb{X}$ be a nonempty set. If $u : \text{Gr}_{\mathbb{X}}(\Phi) \subseteq \mathbb{X} \times \mathbb{Y} \rightarrow \overline{\mathbb{R}}$ is lower semi-continuous at all $(x, y) \in \text{Gr}_Z(\Phi)$ and $\Phi : \mathbb{X} \rightarrow \mathbb{K}(\mathbb{Y})$ is upper semi-continuous at all $x \in Z$, then the function $u(\cdot, \cdot)$ is \mathbb{KN} -inf-compact on $\text{Gr}_Z(\Phi)$;*

Proof The proof is similar to the proof of Feinberg at el. [5, Lemma 3.3(i)]. Let $\{x_i\}_{i \in I}$ be a convergent net with values in \mathbb{X} whose limit x belongs to Z and $\{y_i\}_{i \in I}$ be a net defined on the same ordered set I with $y_i \in \Phi(x_i)$, $i \in I$, and satisfying the condition that the set $\{u(x_i, y_i) : i \in I\}$ is bounded above by $\lambda \in \mathbb{R}$. Let us prove that a net $\{y_i\}_{i \in I}$ has an accumulation point $y \in \Phi(x)$ such that $u(x, y) \leq \lambda$. Aliprantis and Border [1, Corollary 17.17, p. 564] yields that a net $\{y_i\}_{i \in I}$ has an accumulation point $y \in \Phi(x)$. The lower semi-continuity of u at all $(x, y) \in \text{Gr}_Z(\Phi)$ implies that $u(x, y) \leq \lambda$. Therefore, the function $u(\cdot, \cdot)$ is \mathbb{KN} -inf-compact on $\text{Gr}_Z(\Phi)$. \square

When the topological spaces \mathbb{X} and \mathbb{Y} are metrizable, we may avoid nets in the definition of the \mathbb{KN} -inf-compactness by replacing them with sequences.

Lemma 2 *Let \mathbb{X} and \mathbb{Y} be metrizable spaces, $Z \subseteq \mathbb{X}$ be a nonempty set. Then $u : \text{Gr}_{\mathbb{X}}(\Phi) \subseteq \mathbb{X} \times \mathbb{Y} \rightarrow \overline{\mathbb{R}}$ is \mathbb{KN} -inf-compact on $\text{Gr}_Z(\Phi)$ if and only if the following two conditions hold:*

(i) $u : \text{Gr}_{\mathbb{X}}(\Phi) \subseteq \mathbb{X} \times \mathbb{Y} \rightarrow \overline{\mathbb{R}}$ is lower semi-continuous at all $(x, y) \in \text{Gr}_Z(\Phi)$;

(ii) if a sequence $\{x_n\}_{n=1,2,\dots}$ with values in \mathbb{X} converges to $x \in Z$ then each sequence $\{y_n\}_{n=1,2,\dots}$ with $y_n \in \Phi(x_n)$, $n = 1, 2, \dots$, satisfying the condition that the sequence $\{u(x_n, y_n)\}_{n=1,2,\dots}$ is bounded above, has a limit point $y \in \Phi(x)$.

Proof Condition (i) from the definition of \mathbb{KN} -inf-compactness coincides with assumption (i) of the lemma. The rest of the proof establishes the equivalency of assumption (ii) of the lemma and assumption (ii) from Definition 2.

Let assumption (ii) of the lemma hold. Consider a net $\{x_i\}_{i \in I}$ with values in \mathbb{X} , that converges to $x \in Z$, and a net $\{y_i\}_{i \in I}$ defined on the same ordered set I with $y_i \in \Phi(x_i)$, $i \in I$, and satisfying the condition that the set $\{u(x_i, y_i) : i \in I\}$ is bounded above. Let us prove that $\{y_i\}_{i \in I}$ has an accumulation point $y \in \Phi(x)$. Since the space \mathbb{X} is metrizable, it is first-countable and each its point has a countable neighborhood basis (local base). That is, for $x \in Z$ there exists a sequence $\mathcal{O}_1, \mathcal{O}_2, \dots$ of neighborhoods of x such that for each neighborhood $\mathcal{O}(x)$ of x there exists an integer n with \mathcal{O}_n contained in $\mathcal{O}(x)$. Since the net $\{x_i\}_{i \in I}$ converges to x , for each $N = 1, 2, \dots$, there exists an index $i_N \in I$ such that $x_i \in \mathcal{O}_N$ for each $i \succeq i_N$. Thus, the sequence $\{x_{i_N}\}_{N=1,2,\dots}$ converges to x , and the sequence $\{y_{i_N}\}_{N=1,2,\dots}$, with $y_{i_N} \in \Phi(x_{i_N})$, $N = 1, 2, \dots$, satisfies the condition that the set $\{u(x_{i_N}, y_{i_N}) : N = 1, 2, \dots\}$ is bounded above. In view of assumption (ii) of the lemma, the sequence $\{y_{i_N}\}_{N=1,2,\dots}$ has a limit point $y \in \Phi(x)$. Therefore, $y \in \Phi(x)$ is the accumulation point of the net $\{y_i\}_{i \in I}$.

Let assumption (ii) of Definition 2 hold. Consider a sequence $\{x_n\}_{n=1,2,\dots}$ with values in \mathbb{X} , that converges to $x \in Z$, and a sequence $\{y_n\}_{n=1,2,\dots}$ with $y_n \in \Phi(x_n)$, $n = 1, 2, \dots$, such that the sequence $\{u(x_n, y_n)\}_{n=1,2,\dots}$ is bounded above. Let us prove that the sequence $\{y_n\}_{n=1,2,\dots}$ has a limit point $y \in \Phi(x)$. Let C be a closure of the set $\{(x_n, y_n) : n = 1, 2, \dots\}$ in $\mathbb{X} \times \mathbb{Y}$. Condition (ii) of Definition 2 yields that C is a compact set and $C \subseteq \{(x_n, y_n) : n = 1, 2, \dots\} \cup \text{Gr}_{\{x\}}(\Phi)$. Since the space $\mathbb{X} \times \mathbb{Y}$ is metrizable, the sequence $\{(x_n, y_n)\}_{n=1,2,\dots} \subset C$ has a convergent subsequence $\{(x_{n_k}, y_{n_k})\}_{k=1,2,\dots}$ to $(x, y) \in C$. Since $C \subset \text{Gr}_{\mathbb{X}}(\Phi)$, then $(x, y) \in \text{Gr}_{\{x\}}(\Phi) \subseteq \text{Gr}_Z(\Phi)$. Therefore, the sequence $\{y_n\}_{n=1,2,\dots}$ has a limit point $y \in \Phi(x)$. \square

3 Proofs of the Main Results

This section contains the proofs of Theorems 3–7 and Corollary 1.

Proof of Theorem 3. Let $x \in \mathbb{X}$, and the function $u : \text{Gr}_{\mathbb{X}}(\Phi) \subseteq \mathbb{X} \times \mathbb{Y} \rightarrow \overline{\mathbb{R}}$ be \mathbb{KN} -inf-compact on $\text{Gr}_{\{x\}}(\Phi)$. If $v(x) = +\infty$, then $\Phi^*(x) = \Phi(x)$. Otherwise, $\Phi^*(x)$ is a nonempty compact set; cf. Feinberg et al. [5, Theorem 3.1]. Let us

prove that the function $v : \mathbb{X} \rightarrow \overline{\mathbb{R}}$ is lower semi-continuous at x . If $v(x) = -\infty$, then v is lower semi-continuous at x . Let $v(x) > -\infty$. Consider a net $\{x_i\}_{i \in I}$ with values in \mathbb{X} converging to x . Choose a subnet $\{x_j\}_{j \in J}$ of the net $\{x_i\}_{i \in I}$ such that $\liminf_i v(x_i) = \lim_j v(x_j)$. There are two alternatives: either $\{v(x_j)\}_{j \in J}$ converges to $+\infty$, or $\{v(x_j)\}_{j \in J}$ is bounded above. In the first case $v(x) \leq \liminf_i v(x_i) = +\infty$, that is, v is lower semi-continuous at x . If the second alternative holds, then for each $j \in J$ there exists $y_j \in \Phi(x_j)$ such that $\{|v(x_j) - u(x_j, y_j)|\}_{j \in J}$ converges to zero and, therefore, the net $\{u(x_j, y_j)\}_{j \in J}$ is bounded above. Condition (ii) of the definition of \mathbb{KN} -inf-compactness on $\text{Gr}_{\{x\}}(\Phi)$ implies that the net $\{y_j\}_{j \in J}$ has an accumulation point $y \in \Phi(x)$. Condition (i) of the definition of \mathbb{KN} -inf-compactness on $\text{Gr}_{\{x\}}(\Phi)$ yields that $\lim_j v(x_j) \geq u(x, y) \geq v(x)$. Since $\liminf_i v(x_i) = \lim_j v(x_j)$, the function v is lower semi-continuous at x . \square

Proof of Theorem 4. Let $u : \text{Gr}_{\mathbb{X}}(\Phi) \subseteq \mathbb{X} \times \mathbb{Y} \rightarrow \overline{\mathbb{R}}$, $x \in \mathbb{X}$, and a net $\{x_\alpha\}_{\alpha \in I}$ with values in \mathbb{X} converge to x . If $v(x) = +\infty$, then v is obviously upper semi-continuous at x . Consider the case $v(x) < +\infty$.

If assumption (i) holds, then, in view of the lower semi-continuity of Φ at x , for each $y \in \Phi(x)$ and each $\alpha \in I$ there exists $y_\alpha \in \Phi(x_\alpha)$ such that the net $\{y_\alpha\}_{\alpha \in I}$ converges to y in \mathbb{Y} . Therefore, due to upper semi-continuity of $u : \text{Gr}_{\mathbb{X}}(\Phi) \subseteq \mathbb{X} \times \mathbb{Y} \rightarrow \overline{\mathbb{R}}$ at each $(x, y) \in \text{Gr}_{\{x\}}(\Phi)$,

$$\limsup_{\alpha} v(x_\alpha) \leq \limsup_{\alpha} u(x_\alpha, y_\alpha) \leq u(x, y) \text{ for each } y \in \Phi(x),$$

that is,

$$\limsup_{\alpha} v(x_\alpha) \leq \inf_{y \in \Phi(x)} u(x, y) = v(x),$$

and v is upper semi-continuous at x ; Hu and Papageorgiou [9, Proposition 3.1, p. 82].

Let assumption (ii) holds. Denote by $\tau_{\mathbb{X}}$ and $\tau_{\mathbb{Y}}$ the topologies (the families of all open subsets) of \mathbb{X} and \mathbb{Y} respectively. Note that the family of open sets $\tau_{\mathbb{X} \times \mathbb{Y}}^b := \{\mathcal{O}_{\mathbb{X}} \times \mathcal{O}_{\mathbb{Y}} : \mathcal{O}_{\mathbb{X}} \in \tau_{\mathbb{X}}, \mathcal{O}_{\mathbb{Y}} \in \tau_{\mathbb{Y}}\}$ is a base of the topology of the Hausdorff topological product space $\mathbb{X} \times \mathbb{Y}$ (that is, each open subset of $\mathbb{X} \times \mathbb{Y}$ can be presented as a union of some sets from $\tau_{\mathbb{X} \times \mathbb{Y}}^b$). This is true because the product topology (so-called natural or Hausdorff topology) on the Cartesian product of a finite number of Hausdorff spaces coincides with the box topology.

If for any $\lambda > v(x)$ there is a neighborhood of x , $\mathcal{O}(x) \in \tau_{\mathbb{X}}$, such that

$$v(x^*) < \lambda \text{ for each } x^* \in \mathcal{O}(x), \quad (2)$$

then the function v is upper semi-continuous at x . Fix an arbitrary $\lambda > v(x)$. The rest of the proof establishes the existence of $\mathcal{O}(x) \in \tau_{\mathbb{X}}$ satisfying (2).

In view of the upper semi-continuity of $u : \text{Gr}_{\mathbb{X}}(\Phi) \subseteq \mathbb{X} \times \mathbb{Y} \rightarrow \overline{\mathbb{R}}$ at each $(x, y) \in \text{Gr}_{\{x\}}(\Phi^*)$, for each $y \in \Phi^*(x)$ there exists $\mathcal{O}_{\mathbb{X}}^{x,y}(x) \times \mathcal{O}_{\mathbb{Y}}^{x,y}(y) \in \tau_{\mathbb{X} \times \mathbb{Y}}^b$ such that

$$u(x^*, y^*) < \lambda \text{ for all } (x^*, y^*) \in (\mathcal{O}_{\mathbb{X}}^{x,y}(x) \times \mathcal{O}_{\mathbb{Y}}^{x,y}(y)) \cap \text{Gr}_{\mathbb{X}}(\Phi) \quad (3)$$

because $\lambda > v(x) = u(x, y)$ for any $y \in \Phi^*(x)$. The collection of open sets $\{\mathcal{O}_{\mathbb{X}}^{x,y}(x) \times \mathcal{O}_{\mathbb{Y}}^{x,y}(y) : y \in \Phi^*(x)\}$ covers the set $\text{Gr}_{\{x\}}(\Phi^*)$. Tychonoff's theorem yields that the set $\text{Gr}_{\{x\}}(\Phi^*) \subset \mathbb{X} \times \mathbb{Y}$ is compact. Therefore, there is a finite cover $\{\mathcal{O}_{\mathbb{X}}^{x,y_1}(x) \times \mathcal{O}_{\mathbb{Y}}^{x,y_1}(y_1), \mathcal{O}_{\mathbb{X}}^{x,y_2}(x) \times \mathcal{O}_{\mathbb{Y}}^{x,y_2}(y_2), \dots, \mathcal{O}_{\mathbb{X}}^{x,y_N}(x) \times \mathcal{O}_{\mathbb{Y}}^{x,y_N}(y_N)\}$, $y_1, y_2, \dots, y_N \in \Phi^*(x)$, $N = 1, 2, \dots$, of $\text{Gr}_{\{x\}}(\Phi^*)$. Since $x \in \mathcal{O}_{\mathbb{X}}^{x,y_j}(x)$ for each $j = 1, 2, \dots, N$, the family of sets $\{\mathcal{O}_{\mathbb{X}}(x) \times \mathcal{O}_{\mathbb{Y}}^{x,y_1}(y_1), \mathcal{O}_{\mathbb{X}}(x) \times \mathcal{O}_{\mathbb{Y}}^{x,y_2}(y_2), \dots, \mathcal{O}_{\mathbb{X}}(x) \times \mathcal{O}_{\mathbb{Y}}^{x,y_N}(y_N)\}$, where $\mathcal{O}_{\mathbb{X}}(x) = \bigcap_{j=1}^N \mathcal{O}_{\mathbb{X}}^{x,y_j}(x) \in \tau_{\mathbb{X}}$, covers the set $\text{Gr}_{\{x\}}(\Phi^*)$, that is,

$$\text{Gr}_{\{x\}}(\Phi^*) \subseteq \mathcal{O}_{\mathbb{X}}(x) \times \bigcup_{j=1}^N \mathcal{O}_{\mathbb{Y}}^{x,y_j}(y_j) \subseteq \bigcup_{y \in \Phi^*(x)} \mathcal{O}_{\mathbb{X}}^{x,y}(x) \times \mathcal{O}_{\mathbb{Y}}^{x,y}(y). \quad (4)$$

In particular, $\mathcal{O}_{\mathbb{Y}}(\Phi^*(x)) := \bigcup_{j=1}^N \mathcal{O}_{\mathbb{Y}}^{x,y_j}(y_j) \in \tau_{\mathbb{Y}}$ is a neighborhood of $\Phi^*(x)$. Assumption (ii) implies that for the neighborhood $\mathcal{O}_{\mathbb{Y}}(\Phi^*(x))$ of $\Phi^*(x)$ there exists a neighborhood $\mathcal{O}_{\mathbb{X}}^1(x)$ of x such that $\mathcal{O}_{\mathbb{Y}}(\Phi^*(x)) \cap \Phi(x^*) \neq \emptyset$ for all $x^* \in \mathcal{O}_{\mathbb{X}}^1(x)$. Therefore, according to (3) and (4),

$$v(x^*) \leq \inf_{y^* \in \mathcal{O}_{\mathbb{Y}}(\Phi^*(x)) \cap \Phi(x^*)} u(x^*, y^*) < \lambda,$$

for each $x^* \in \mathcal{O}(x) := \mathcal{O}_{\mathbb{X}}(x) \cap \mathcal{O}_{\mathbb{X}}^1(x) \in \tau_{\mathbb{X}}$. Thus (2) holds. \square

Proof of Theorem 5. Let $u : \text{Gr}_{\mathbb{X}}(\Phi) \subseteq \mathbb{X} \times \mathbb{Y} \rightarrow \overline{\mathbb{R}}$, $x \in \mathbb{X}$, and let $v : \mathbb{X} \rightarrow \overline{\mathbb{R}}$ be the value function defined in (1). If u is \mathbb{KN} -inf-compact on $\text{Gr}_{\{x\}}(\Phi)$, v is continuous at x , and $v(x) < +\infty$, then Theorem 3 yields that $\Phi^*(x)$ is a nonempty compact set. Let us prove that Φ^* is upper semi-continuous at x . Suppose, on the contrary, that Φ^* is not upper semi-continuous at $x \in \mathbb{X}$. Then there is an open neighborhood $\mathcal{O}(\Phi^*(x))$ of $\Phi^*(x)$ such that, for every neighborhood \mathcal{O} of x , there is an $x_{\mathcal{O}} \in \mathcal{O}$ with $\Phi^*(x_{\mathcal{O}}) \not\subseteq \mathcal{O}(\Phi^*(x))$. Therefore, there exists $y_{\mathcal{O}} \in \Phi^*(x_{\mathcal{O}}) \setminus \mathcal{O}(\Phi^*(x))$. Now consider the nets $\{x_{\mathcal{O}} : \mathcal{O} \in \tau_b^{\text{loc}}(x)\}$ and $\{y_{\mathcal{O}} : \mathcal{O} \in \tau_b^{\text{loc}}(x)\}$, where $\tau_b^{\text{loc}}(x)$ is the directed set of neighborhoods of x ; see Zgurovsky et al. [12, p. 9] for details. The net $\{x_{\mathcal{O}} : \mathcal{O} \in \tau_b^{\text{loc}}(x)\}$ converges to x . Then each net $\{y_{\mathcal{O}} : \mathcal{O} \in \tau_b^{\text{loc}}(x)\}$, defined on the same ordered set $\tau_b^{\text{loc}}(x)$ with $y_{\mathcal{O}} \in \Phi^*(x_{\mathcal{O}})$, $\mathcal{O} \in \tau_b^{\text{loc}}(x)$, has an accumulation point $y \in \Phi^*(x)$. Indeed, $v(x_{\mathcal{O}}) = u(x_{\mathcal{O}}, y_{\mathcal{O}})$ for each $\mathcal{O} \in \tau_b^{\text{loc}}(x)$. Since $x_{\mathcal{O}} \rightarrow x$ and v is continuous at x , the net $\{v(x_{\mathcal{O}}) : \mathcal{O} \in \tau_b^{\text{loc}}(x)\}$ is bounded above by a finite constant eventually in $\tau_b^{\text{loc}}(x)$. Therefore, \mathbb{KN} -inf-compactness of the function u on $\text{Gr}_{\{x\}}(\Phi)$ implies that the net $\{y_{\mathcal{O}} : \mathcal{O} \in \tau_b^{\text{loc}}(x)\}$ has an accumulation point $y \in \Phi(x)$. Since $u : \text{Gr}_{\mathbb{X}}(\Phi) \subseteq \mathbb{X} \times \mathbb{Y} \rightarrow \overline{\mathbb{R}}$ is lower semi-continuous at each $(x, y) \in \text{Gr}_{\{x\}}(\Phi)$ and v is continuous at x , then $y \in \Phi^*(x)$, that is, the net $\{y_{\mathcal{O}} : \mathcal{O} \in \tau_b^{\text{loc}}(x)\}$ has an accumulation point $y \in \Phi^*(x) \subseteq \mathcal{O}(\Phi^*(x))$. The net $\{y_{\mathcal{O}} : \mathcal{O} \in \tau_b^{\text{loc}}(x)\}$ lies in the closed set $\mathcal{O}(\Phi^*(x))^c$, which is the complement of $\mathcal{O}(\Phi^*(x))$, and thus $y \in \mathcal{O}(\Phi^*(x))^c$. This contradiction implies that Φ^* is upper semi-continuous at x . \square

Proof of Theorem 6. According to Theorem 3, the function $v : \mathbb{X} \rightarrow \overline{\mathbb{R}}$ is lower semi-continuous at x and $\Phi^*(x)$ is a nonempty compact set. If assumptions (i) and (ii) of Theorem 6 imply each other, then Theorem 4(ii) yields that the value function v is upper semi-continuous at x . Therefore, to finish the proof, it is sufficient to prove that assumptions (i) and (ii) of Theorem 6 are equivalent.

The implication “(i) \implies (ii)” directly follows from Theorems 3, 4(ii), and 5. Vice versa, if the solution multifunction $\Phi^* : \mathbb{X} \rightarrow 2^{\mathbb{Y}}$ is upper semi-continuous at x , then for each neighborhood $\mathcal{O}(\Phi^*(x))$ of the set $\Phi^*(x)$, there is a neighborhood of x , say $\mathcal{O}(x)$, such that $\Phi^*(x^*) \subseteq \mathcal{O}(\Phi^*(x))$ for all $x^* \in \mathcal{O}(x)$. Thus, for each neighborhood $\mathcal{O}(\Phi^*(x))$ of the set $\Phi^*(x)$ there exists a neighborhood $\mathcal{O}(x)$ of x such that $\mathcal{O}(\Phi^*(x)) \cap \Phi(x^*) \neq \emptyset$ for all $x^* \in \mathcal{O}(x)$. Therefore, assumptions (i) and (ii) of Theorem 6 are equivalent. \square

Proof of Corollary 1. According to Theorem 3, the function $v_{\lambda,x} : \mathbb{X} \rightarrow \overline{\mathbb{R}}$ is lower semi-continuous at x and $\Phi_{\lambda,x}^*(x)$ is a nonempty compact set because $v_{\lambda,x}(x) \leq \lambda < +\infty$. Since $u(x, y) \leq \lambda$ for some $y \in \Phi(x)$, then $y \in \Phi^*(x)$ and $v(x) = v_{\lambda,x}(x)$. Moreover, if $\Phi_{\lambda,x}^*(x^*) \neq \emptyset$ for some $x^* \in \mathbb{X}$, then $v_{\lambda,x}(x^*) = v(x^*)$; otherwise, $v(x^*) \geq \lambda \geq v(x)$. Therefore, the function $v : \mathbb{X} \rightarrow \overline{\mathbb{R}}$ is lower semi-continuous at x and $\Phi^*(x)$ is a nonempty compact set. \square

Proof of Theorem 7. According to Corollary 1, the value function v is lower semi-continuous at x and $\Phi^*(x) \in \mathbb{K}(\mathbb{Y})$. Since $u(x, y) < \lambda$ for some $y \in \Phi(x)$ and the function $u : \text{Gr}_{\mathbb{X}}(\Phi) \subseteq \mathbb{X} \times \mathbb{Y} \rightarrow \overline{\mathbb{R}}$ is upper semi-continuous at all $(x, y) \in \text{Gr}_{\{x\}}(\Phi^*)$, there exists a neighborhood of x , say $\mathcal{O}(x)$, such that the level set $\mathcal{D}_{u(x^*, \cdot)}(\lambda; \Phi(x^*))$ is nonempty for each $x^* \in \mathcal{O}(x)$. Therefore, $v_{\lambda,x}(x^*) = v(x^*)$ and $\Phi_{\lambda,x}^*(x^*) = \Phi^*(x^*)$ for each $x^* \in \mathcal{O}(x)$. Note that assumption (ii) of Theorem 7 implies assumption (i) of Theorem 7. Theorem 4 yields that the function $v : \mathbb{X} \rightarrow \overline{\mathbb{R}}$ is upper semi-continuous at x . Therefore, the function $v = v_{\lambda,x}$ is continuous at x . Being applied to $u_{\lambda,x}$, $v_{\lambda,x}$ and $\Phi_{\lambda,x}$, Theorem 5, yields that assumptions (i) and (ii) of Theorem 7 are equivalent. \square

4 Examples and Counterexamples

The following example illustrates that Theorem 2 can be applied to a function $u : \mathbb{X} \times \mathbb{Y} \rightarrow \mathbb{R}$, which is not \mathbb{KN} -inf-compact on $\text{Gr}_{\mathbb{X}}(\Phi)$.

Example 1 Let $\mathbb{X} = \mathbb{Y} = \mathbb{R}$, $u(x^*, y^*) = \min\{|x^* - y^*|, 1\}$, $\Phi(x^*) = \mathbb{R}$, $x^*, y^* \in \mathbb{R}$. Then u is neither \mathbb{K} -inf-compact nor \mathbb{KN} -inf-compact on \mathbb{R}^2 , because $\mathcal{D}_u(1; \text{Gr}_{\{0\}}(\Phi)) = \mathbb{Y} \notin \mathbb{K}(\mathbb{Y})$. Thus, the assumptions of Theorem 1 do not hold, but the assumptions of Theorem 2 are obviously hold for $x = 0$, $\lambda = \frac{1}{2}$, and $\Phi^*(0) = \{0\}$. \square

In the following example the assumptions of Theorem 1 hold, but the assumptions of Theorem 2 do not hold.

Example 2 Let $\mathbb{X} = \mathbb{Y} = l_1 := \{(a_1, a_2, \dots) : |a_1| + |a_2| + \dots < \infty\}$ be metric spaces with the metric $\rho(a, b) := \|a - b\| = |a_1 - b_1| + |a_2 - b_2| + \dots$ for each $a = (a_1, a_2, \dots)$, $b = (b_1, b_2, \dots) \in l_1$. Denote $\bar{0} := (0, 0, \dots)$ and $\bar{B}_\delta(\bar{0}) = \{a = (a_1, a_2, \dots) \in l_1 : |a_1| + |a_2| + \dots \leq \delta\}$, where $\delta > 0$. The balls $\bar{B}_\delta(\bar{0})$ are not compact sets because the sequence $\{a^{(n)}\}_{n=1,2,\dots}$, with $a_n^{(n)} = \delta$ and $a_i^{(n)} = 0$ for $i \neq n$, does not contain a convergent subsequence.

Let $\Phi(x^*) = \{x^*\}$ and $u(x^*, y^*) = \frac{1}{2}(\|x^*\| + \|y^*\|)$ for all $(x^*, y^*) \in \mathbb{X} \times \mathbb{Y}$. Fix $x = \bar{0}$. Note that u is continuous on $\mathbb{X} \times \mathbb{Y}$ and \mathbb{KN} -inf-compact on $\text{Gr}_{\mathbb{X}}(\Phi) = \{(z, z) : z \in \mathbb{X}\}$. The set-valued mapping $\Phi : \mathbb{X} \rightarrow \mathbb{K}(\mathbb{Y})$ is continuous. Therefore, the assumptions of Theorem 1 hold.

Assumption (iii) of Theorem 2 does not hold. Indeed, on the contrary, if there exist $\lambda \in \mathbb{R}$ and a compact set $C \subset \mathbb{Y}$ such that, for every x^* in a neighborhood of $\bar{0}$, say $\mathcal{O}(\bar{0})$, the level set $\mathcal{D}_{u(x^*, \cdot)}(\lambda; \Phi(x^*))$ is nonempty and contained in C , then there exists $\delta > 0$ such that $\mathcal{O}(\bar{0}) \supseteq \bar{B}_\delta(\bar{0})$, $\delta \leq \lambda$, and $\bar{B}_\delta(\bar{0}) \subseteq C$. Since the closed subset $\bar{B}_\delta(\bar{0})$ of a compact set C is compact, we obtain a contradiction, because the ball $\bar{B}_\delta(\bar{0})$ is not compact. \square

The following example demonstrates that (a): assumptions (i) and (ii) of Theorem 4 do not imply each other; and (b) statements (B2) and (BS2) do not imply each other.

Example 3 Set $\mathbb{X} = [0, 1]$, $\mathbb{Y} = [-1, 0]$, $u(x^*, y^*) = |x^* - y^*|$, $(x^*, y^*) \in \mathbb{X} \times \mathbb{Y}$, and $x = 0$. Note that u is finite and continuous on $\mathbb{X} \times \mathbb{Y}$.

Let $\Phi(x^*) = [-1, 0]$, $x^* \in \mathbb{X}$. Then $\Phi : \mathbb{X} \rightarrow \mathbb{S}(\mathbb{Y})$ is a lower semi-continuous set-valued mapping. Thus assumption (i) of Theorem 4 and the assumptions of statement (B2) hold. However, neither assumption (ii) of Theorem 4 nor the assumptions of statement (BS2) hold, because $\Phi(x^*) = \emptyset$ for each $x^* \in \mathbb{X}$.

Now let $\Phi(x^*) = \{0, -\mathbf{I}\{x^* = 0\}\}$, $x^* \in \mathbb{X}$. Then $\Phi^*(x^*) = \{0\} \in \mathbb{K}(\mathbb{Y})$, for each $x^* \in \mathbb{X}$, and for each neighborhood $\mathcal{O}(\Phi^*(x))$ of the set $\Phi^*(x)$ there exists a neighborhood $\mathcal{O}(x)$ of x such that $\mathcal{O}(\Phi^*(x)) \cap \Phi(x^*) \neq \emptyset$ for all $x^* \in \mathcal{O}(x)$. Thus assumption (ii) of Theorem 4 and the assumptions of statement (BS2) hold. However, neither assumption (i) of Theorem 4 nor the assumptions of statement (B2) hold, because Φ is not lower semi-continuous at x . \square

The following two examples yield that statements (B3) and (BS3) do not imply each other.

Example 4 (The assumptions of statement (BS3) hold, but the assumptions of statement (B3) do not hold). For this purpose, consider Example 1. Note that: (a) the function u is continuous on $\mathbb{X} \times \mathbb{Y}$, (b) the set-valued mapping Φ is closed, (c) for every x^* in a neighborhood of $x = 0$, say $(-\frac{1}{2}, \frac{1}{2})$, the level set $\mathcal{D}_{u(x^*, \cdot)}(\lambda; \Phi(x^*)) = [x^* - \frac{1}{2}, x^* + \frac{1}{2}]$ is nonempty and is contained in the compact set $C = [-1, 1]$, and (d) for each neighborhood $\mathcal{O}(\Phi^*(x))$ of the set $\Phi^*(x) = \{0\}$ there exists a neighborhood $\mathcal{O}(x) = (-1, 1)$ of x such that $\mathcal{O}(\Phi^*(x)) \cap \Phi(x^*) \neq \emptyset$ for all $x^* \in \mathcal{O}(x)$. The latter is true since $\Phi \equiv \mathbb{R}$. Therefore, the assumptions of statement (BS3) hold, but the assumptions of statement (B3) do not hold. \square

Example 5 (The assumptions of statement (B3) hold, but the assumptions of statement (BS3) do not hold). Let $\mathbb{X} = [0, 1]$, $\mathbb{Y} = [0, 2]$, $\Phi \equiv [0, 2]$, $u(x^*, y^*) = \mathbf{I}\{x^* - y^* < 0\}$, $x^* \in \mathbb{X}$, $y^* \in \mathbb{Y}$. Then u is \mathbb{KN} -inf-compact on $\mathbb{X} \times \mathbb{Y}$, $v \equiv 0$ is continuous, but u is discontinuous at each point $(x, x) \in [0, 1]^2$. Therefore, the assumptions of statement (B3) hold, but the assumptions of statement (BS3) do not hold. \square

The following example shows that \mathbb{KN} -inf-compactness of u on $\text{Gr}_{\{x\}}(\Phi)$ in Theorem 5 cannot be weakened to lower semi-continuity of $u : \text{Gr}_{\mathbb{X}}(\Phi) \subseteq \mathbb{X} \times \mathbb{Y} \rightarrow \mathbb{R}$, inf-compactness of $u(x, \cdot)$ on $\Phi(x)$, and closeness of $\text{Gr}_{\mathbb{X}}(\Phi)$ in $\mathbb{X} \times \mathbb{Y}$ with $\Phi : \mathbb{X} \rightarrow \mathbb{K}(\mathbb{Y})$.

Example 6 Let $\mathbb{X} = [0, 1]$, $\mathbb{Y} = [0, +\infty)$, $u \equiv 0$, $x = 0$, $\Phi(x^*) = \{1/x^*\}$ if $x^* \in (0, 1]$, and $\Phi(0) = \{0\}$. Then u is continuous on $\mathbb{X} \times \mathbb{Y}$, $v \equiv 0$ is continuous on \mathbb{X} , $\Phi^*(x^*) = \Phi(x^*) \in \mathbb{K}(\mathbb{Y})$, $x^* \in \mathbb{X}$, $\text{Gr}_{\mathbb{X}}(\Phi)$ is closed in $\mathbb{X} \times \mathbb{Y}$, $u(0, \cdot) \equiv 0$ is inf-compact on $\Phi^*(x) = \{0\}$, but Φ^* is not upper semi-continuous at $x = 0$. \square

The following example demonstrates that continuity of v at x in Theorem 5 is the essential assumption.

Example 7 Let $\mathbb{X} = \mathbb{Y} = [0, 1]$, $u(x^*, y^*) = y^* \mathbf{I}\{x^* > 0\}$, $x^*, y^* \in [0, 1]$, $\Phi(x^*) = \{1/x^*\}$ for $x^* \in (0, 1]$, $\Phi(0) = \{0\}$, and $x = 0$. Then, u is \mathbb{KN} -inf-compact on $\text{Gr}_{\mathbb{X}}(\Phi)$, $v(x^*) = \frac{1}{x^*}$ for $x^* \in (0, 1]$, $v(0) = 0$, and $\Phi^*(x^*) = \Phi(x^*)$, $x^* \in \mathbb{X}$. Thus, v is discontinuous at x and $\Phi^* : \mathbb{X} \rightarrow \mathbb{K}(\mathbb{Y})$ is not upper semi-continuous at x . \square

In the following example the assumptions of Corollary 7 hold, but the assumptions of Theorem 2 do not hold.

Example 8 Let $\mathbb{X} = \mathbb{Y} = [0, 1]$, $\Phi(x^*) = \{x^*\}$ for all $x^* \in [0, 1]$, and $u(x^*, y^*) = -x^* \mathbf{I}\{x^* \in \mathbb{Q}\}$ for $x^*, y^* \in [0, 1]$, where \mathbb{Q} is the set of rational numbers. Assumptions of Corollary 7 hold for $x = 0$ and $\lambda = 0$, because $\Phi_{\lambda, x} = \Phi$ is continuous, $u_{\lambda, x} = u$ is \mathbb{KN} -inf-compact on $\text{Gr}_{\{0\}}(\Phi_{\lambda, x}) = \{(0, 0)\}$ and it is continuous at the point $(0, 0)$. Assumption (i) of Theorem 2 does not hold, because the function $x^* \rightarrow -x^* \mathbf{I}\{x^* \in \mathbb{Q}\}$ is not continuous on $[0, a]$ for each $a \in (0, 1)$. \square

The following example demonstrates that upper semi-continuity of u at all $(x, y) \in \text{Gr}_{\{x\}}(\Phi^*)$ in Corollary 7 cannot be relaxed to upper semi-continuity of $u_{\lambda, x}$ at all $(x, y) \in \text{Gr}_{\{x\}}(\Phi^*)$.

Example 9 Let $\mathbb{X} = \mathbb{Y} = [0, 1]$, $\Phi(x^*) = \{x^*\}$ and $u(x^*, y^*) = \mathbf{I}\{x^* \neq 0\}$ for all $x^* \in [0, 1]$ and $y^* = x^*$, $\lambda = \frac{1}{2}$, and $x = 0$. Then $u(0, 0) < \lambda$, the function $u_{\lambda, x} \equiv 0$ is \mathbb{KN} -inf-compact on $\text{Gr}_{\{x\}}(\Phi_{\lambda, x})$ and upper semi-continuous at any $(x, y) \in \text{Gr}_{\mathbb{X}}(\Phi)$, but the function u is not upper semi-continuous at $(0, 0) \in \text{Gr}_{\{0\}}(\Phi^*)$ and the value function $v(x^*) = \mathbf{I}\{x^* \neq 0\}$, $x^* \in [0, 1]$, is not continuous at $x = 0$. \square

Acknowledgements The authors thank Alexander Shapiro for bringing the local maximum theorem from Bonnans and Shapiro [3, Proposition 4.4] to their attention and for stimulating discussions.

References

1. C.D. Aliprantis, K.C. Border, *Infinite Dimensional Analysis: A Hitchhiker's Guide*, third ed., xxii+704 pp. Springer Verlag, Berlin (2006)
2. C. Berge, *Topological Spaces*, xiii+270 pp. Macmillan, New York (1963)
3. J.F., Bonnans and A. Shapiro, *Perturbation Analysis of Optimization Problems*, xviii+601 pp. Springer, New York (2000)
4. R. Engelking, *General Topology*, revised ed., 540 pp. Helderman Verlag, Berlin, (1989)
5. E.A. Feinberg, P.O. Kasyanov, M. Voorneveld, Berge's maximum theorem for noncompact image sets, *J. Math. Anal. Appl.*, 413, pp. 1040–1046 (2014)
6. E.A. Feinberg, P.O. Kasyanov, N.V. Zadoianchuk, Average cost Markov decision processes with weakly continuous transition probabilities, *Math. Oper. Res.*, 37(4), pp. 591–607 (2012) .
7. E.A. Feinberg, P.O. Kasyanov, N.V. Zadoianchuk, Berge's theorem for noncompact image sets, *J. Math. Anal. Appl.*, 397(1), pp. 255–259 (2013)
8. E.A. Feinberg, M. E. Lewis, Optimality inequalities for average cost Markov decision processes and the stochastic cash balance problem, *Math. Oper. Res.* 32(4), pp. 769–783 (2007)
9. Sh. Hu and N.S. Papageorgiou, *Handbook of Multivalued Analysis. Volume I: Theory*, xxx+968 pp. Kluwer, Dordrecht (1997)
10. J.L. Kelley, *General Topology*, 298 pp. Van Nostrand, New York (1955)
11. J.R. Munkres, *Topology*, second ed., 537 pp. Prentice Hall, New York (2000).
12. M.Z. Zgurovsky, V.S. Mel'nik, P.O. Kasyanov, *Evolution Inclusions and Variation Inequalities for Earth Data Processing I*, xxx+250 pp. Springer, Berlin (2011)

